Chapter 1

Motion due to an inverse square attractive force

1.1 Introduction

These notes are designed to apply basic geometry and the conservation of energy to derive the basic equations of orbital motion starting from Newton's theory of gravity. It is based in large part on Feynman's approach as described in Goodstein and Goodstein [1]. We shall do more with energy conservation and other analytical details because this will probably help with your research and connect better with modern notation. Maxwell [2] achieves much the same effect without explicitly invoking energy conservation. To do so, however, he must deftly apply arcane properties of the ellipse which makes the approach somewhat more opaque. For our purposes, it is better to use energy conservation and avail ourselves of the simplicity of algebra over geometry at the end of the sotry. For definiteness we shall frame the discussion in terms of an asteroid orbiting the Sun.

1.2 Central Forces

Newton's gravitational force always pulls the orbiting object towards a fixed point (for example, the Sun). A force always directed towards a fixed point is called a central force. Here we show that central forces have an interesting property regardless of how they depend on distance from the point of attraction.

We shall suppose that the force is delivered to the asteroid as series of discrete whacks or impulses. Each impulse is delivered over an extremely short interval of time (effectively a single moment). This means that the orbiting body moves in straight line segments puncuated by small but finite changes in velocity. For continuous forces we may imagine the impulses to grow smaller in magnitude and greater in number, ultimatley becoming an infinite number of infinitesmal impulses. For now we stick to the finite case.

The basic idea is illustrated in figure 1.1. We begin following our asteroid when it is at point A. We suppose that it travels (force free) to point B in one unit of time. This can be written

$$\overrightarrow{AB} = \mathbf{v} \times \text{unit time} \tag{1.1}$$

If the asteroid were to continue unmolested by any impulse then after one more unit of time the asteroid would find itself at point C. If instead upon reaching B the asteroid suffers an impulse directed towards the Sun (point S) then its velocity will be changed by a vector amount $\Delta\mathbf{v}$ which points in the direction from B to S. Let the directed line segment $\overrightarrow{BB'}$ be the displacement of the asteroid in unit time due to the impulse. Formally we may write

$$\overrightarrow{BB'} = \Delta\mathbf{v} \times \text{unit time} \tag{1.2}$$

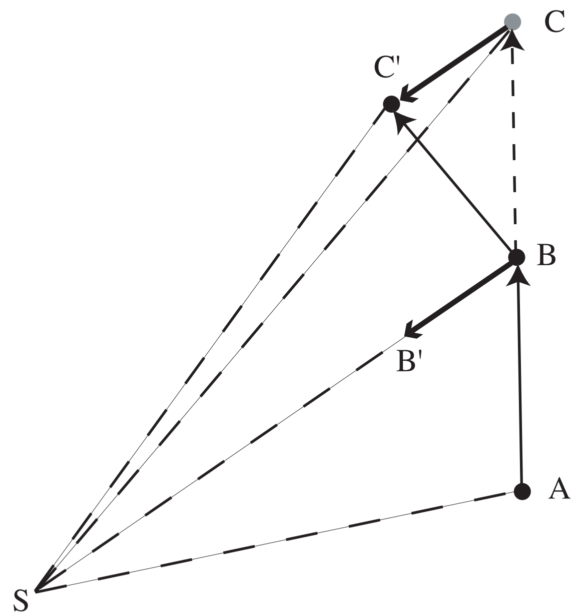

Figure 1.1: We get our asteroid kicked at B.

In order to find where the asteroid ends up one unit of time after the impulse we must simply add the displacement $\overrightarrow{BB'}$ to the position C. We can do this geometrically as shown in figure 1.1 by parallel transporting $\overrightarrow{BB'}$ to form $\overrightarrow{CC'}$.

Now comes very important reasoning. Since $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ we may conclude

$$\text{area of } \triangle ASB = \text{area of } \triangle BSC \tag{1.3}$$

Since $\overrightarrow{CC'}$ is parallel to $\overrightarrow{BB'}$ we may conclude

$$\text{area of } \triangle BSC = \text{area of } \triangle BSC' \tag{1.4}$$

Therefore we have

$$\text{area of } \triangle ASB = \text{area of } \triangle BSC' \tag{1.5}$$

Note that this is an exact statment, not dependent on making the unit time interval very small.

Since the time interval from $\overrightarrow{AB}$ is equal to the time interval from $\overrightarrow{BC'}$, we have shown that the rate at which the orbiting asteroid sweeps out area is constant throughout the orbit so long as the force (or kicks) are always directed toward S. This statement is called Kepler's 2^{nd} law. Traditionally we define

$$h \equiv 2 \times \text{rate of area sweep} \tag{1.6}$$

1.3 Polar Coordinates

In describing the motion due to a central force we often use polar coordinates as depicted in figure 1.2.

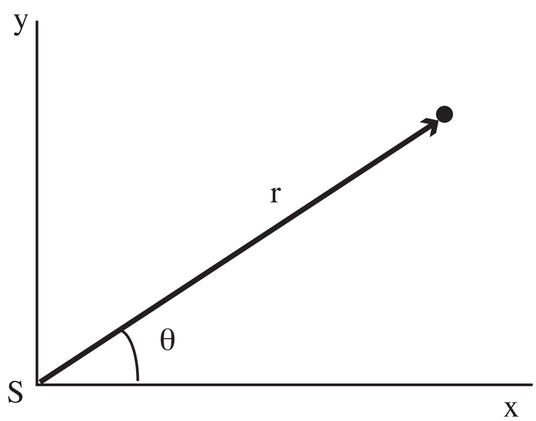

Figure 1.2: Definition of polar coordinates.

It is useful to know h in terms of these coordinates. Let us imagine that during a very small time interval the asteroid moves from a point A to a point B as depicted in figure 1.3.

The area swept out ($\triangle SAB$) can be approximated

$$\text{area of } \triangle SAB \approx \text{area of } \triangle SAP \approx \frac{1}{2}(r\Delta\theta) \times r \tag{1.7}$$

Dividing this equation by the time interval Δt corresponding to the displacement $\overrightarrow{AB}$ we may write equation 1.6 as

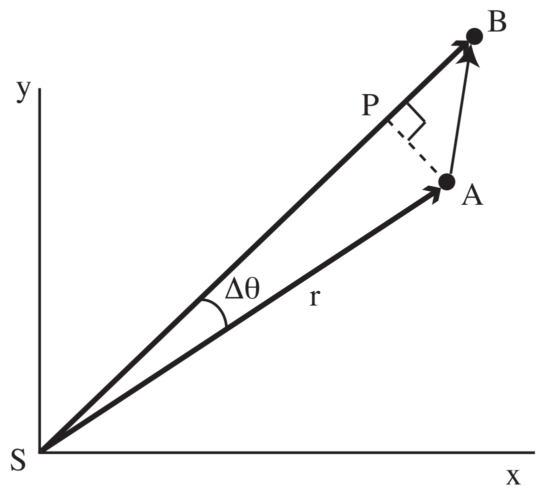

Figure 1.3: Calculating h in polar coordinates.

$$h = r^2 \frac{\Delta\theta}{\Delta t} \tag{1.8}$$

To reiterate, this quantity is constant for any central force orbit.

1.4 Finding the motion

First we note that for a bounded orbit there will be a point of closest approach, called the perihelion, and farthest approach, called the aphelion. We orient the orbit so that the perihelion is along the $+x$ axis. Figure 1.4 shows the perihelion at position r_p. Note that the velocity of the asteroid at perihelion must be perpendicular to the position vector at perihelion, $\mathbf{r}_p \perp \mathbf{v}_p$. Likewise, at aphelion $\mathbf{r}_a \perp \mathbf{v}_a$.

The key to the geometric approach here is to imagine that the orbit in space discretized not into displacements equal in time intervals but rather equal in angle intervals as shown in figure 1.5.

Equation 1.8 gives us Δt:

$$\Delta t = \frac{r^2}{h} \Delta\theta \tag{1.9}$$

For each angle increment there is a velocity change Δv which has magnitude

$$|\Delta v| = \frac{\mu}{r^2}\Delta t = \frac{\mu}{r^2} \times \frac{r^2}{h}\Delta\theta = \frac{\mu\Delta\theta}{h} \tag{1.10}$$

It is useful to visualize all the $\Delta\mathbf{v}$ vectors arranged with tails together as shown in figure 1.6. This configuration makes it clear that if we add all the

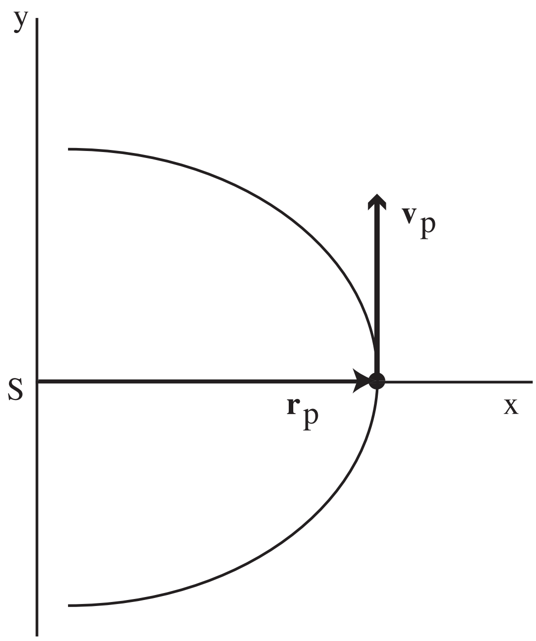

Figure 1.4: Standard orientation of orbit.

$\Delta\mathbf{v}$ vectors of an orbit together we get the zero vector. Note well that this diagram looks the same whether or not the orbit is circular.

We now imagine adding the vectors head–to–tail style. Since the sum is the zero vector the and all $\Delta\mathbf{v}$ vectors have the same the result must be an equilateral polygon. In the limit of vanishing $\Delta\theta$ the polygon becomes a circle. Again, note that the head–to–tail arrangement of the $\Delta\mathbf{v}$ vectors form a circle even for non–circular orbits. However, the center of the velocity circle represents the zero velocity only if the orbit is circular. Such a diagram in velocity space is called a hodogram. Figure 1.7 shows the velocity circle and the orbit in space. The orientation is arranged so that the velocity directions are consistent with the corresponding diagram of the orbit in coordinate space. Notice, for example, that the perihelion vector $\mathbf{v}_p$ is pointing upward just as in figure 1.4. The radius of the velocity circle is $V = (v_p + v_a)/2$

Note that the hodograph provides a graphical demonstration that the orbit is closed. We also see that $\mathbf{v}_p$ and $\mathbf{v}_a$ are in opposite directions and 180° apart in the orbit. We also note that Kepler's Second Law implies

$$h = v_a r_a = v_p r_p \tag{1.11}$$

This expression will be used later.

1.5 Showing that the orbit is an ellipse

We now redraw the hodograph of figure 1.7 rotated clockwise by 90°. This will enable us to use the hodograph to draw a curve whose tangent vector

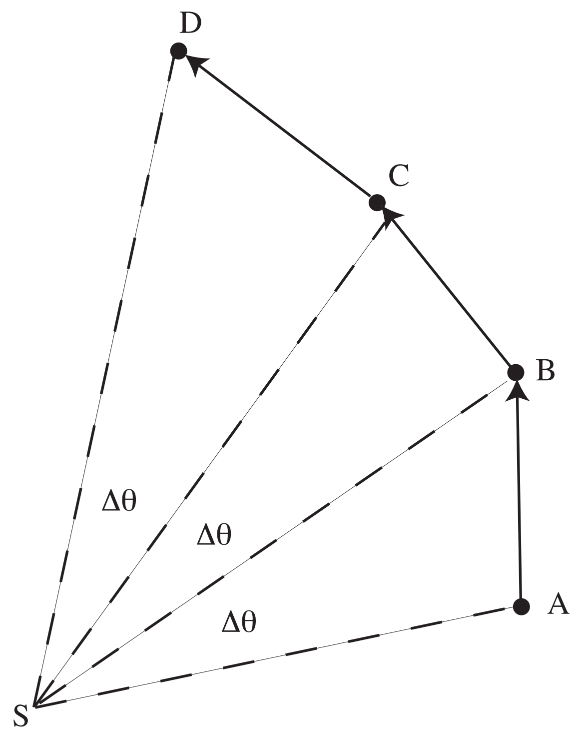

Figure 1.5: Orbit divided into $\Delta\theta = 12°$ intervals.

is at all points parallel to the velocity vectors **v** in the original hodograph in figure 1.7. We shall show that the curve which satisfies this condition is indeed an ellipse. But the velocity of the asteroid is, by definition, tangent to its own path through space: the curve we construct is the orbit.

Figure 1.8 shows one half of the velocity circle. Note that the velocity vector in figure 1.8 is perpendicular to the velocity vector in figure 1.7. Now we construct a curve whose tangent is parallel to the velocity vector in figure 1.7.

Consider an arbitrary velocity vector **v** designated $\overrightarrow{OQ}$ in figure 1.9. We first draw a perpendicular bisector to the segment $\overrightarrow{OQ}$ indicated by the dotted line in figure 1.9. This dotted line is perpendicular to the perpendicular to the actual velocity vector **v** in figure 1.7. Therefore, the dotted line must be parallel to the actual velocity vector **v** in figure 1.7.

We consider the point (labeled I in figure 1.9) marking the intersection of the perpendicular bisector to $\overrightarrow{OQ}$ with the velocity circle radius vector $\mathbf{V} = \overrightarrow{CQ}$. By similar triangles (sideangle–side) we have $\overline{OI} = \overline{IQ}$. We also know that $\overline{CI} + \overline{IQ} = \overline{CQ}$ where $\overline{CQ}$ is the *constant* radius, V, of the velocity circle. Replacing $\overline{IQ}$ with $\overline{OI}$ we have

$$\overline{CI} + \overline{OI} = \overline{CQ} \tag{1.12}$$

This equation holds as we apply this construction to all velocity vectors **v** on figure 1.9. As we continuously change $\mathbf{v} = \overrightarrow{OQ}$ the successive points

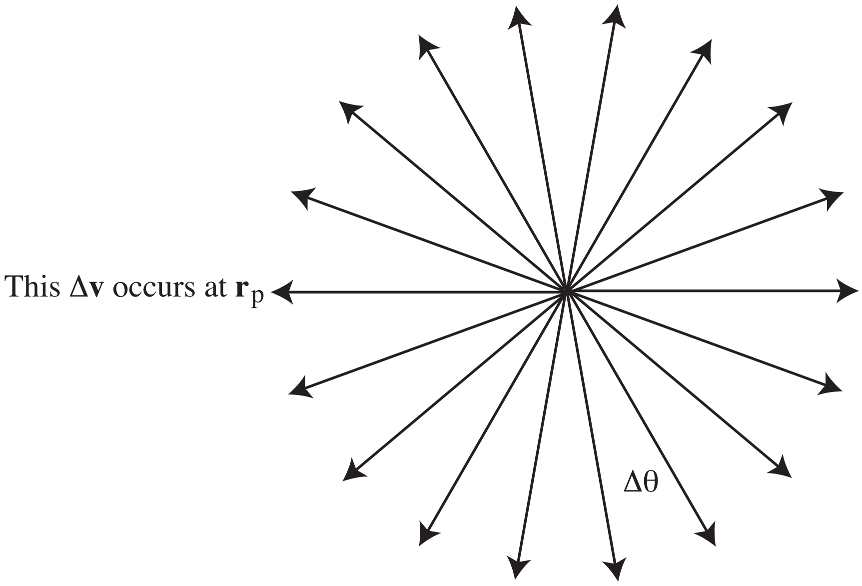

Figure 1.6: $\Delta\mathbf{v}$ vectors for orbit divided into $\Delta\theta = 20°$ intervals.

I trace a continuous path. Equation 1.12 holds for all points I on this path. This equation amounts to the familiar definition of an ellipse based on construction using a string of length $\overline{CQ}$. Thus, we have shown that the points I form an ellipse. This does not prove that it is the orbit. For that we only need to establish that the perpendicular bisector is the tangent to the path at each point I. This follows easily from the fact that the points CIQ lie on a straight line. For this implies that if we consider any other point I' on the perpendicular bisector we have

$$\overline{CI'} + \overline{OI'} > \overline{CI} + \overline{OI} \tag{1.13}$$

so that I' is a point on a bigger ellipse. Thus, the point I' is a point on a bigger ellipse. We conclude that the perpendicular bisectors are each tangent to the curve traced by each point I. The curve traced by successive points I is indeed the orbit of the asteroid corresponding to the velocity circle. The orbit of the asteroid is an ellipse. We show some ellipse terminology in figure 1.10.

Of course, the scale of our ellipse is not determined since it has been constructed in velocity space. We may use energy conservation to connect velocity space to coordinate space and derive an analytic equation describing the orbit.

1.6 Implications of energy conservation

We have established that the orbit is an ellipse. We establish certain traditional definitions describing the size and shape of an ellipse. A little thought

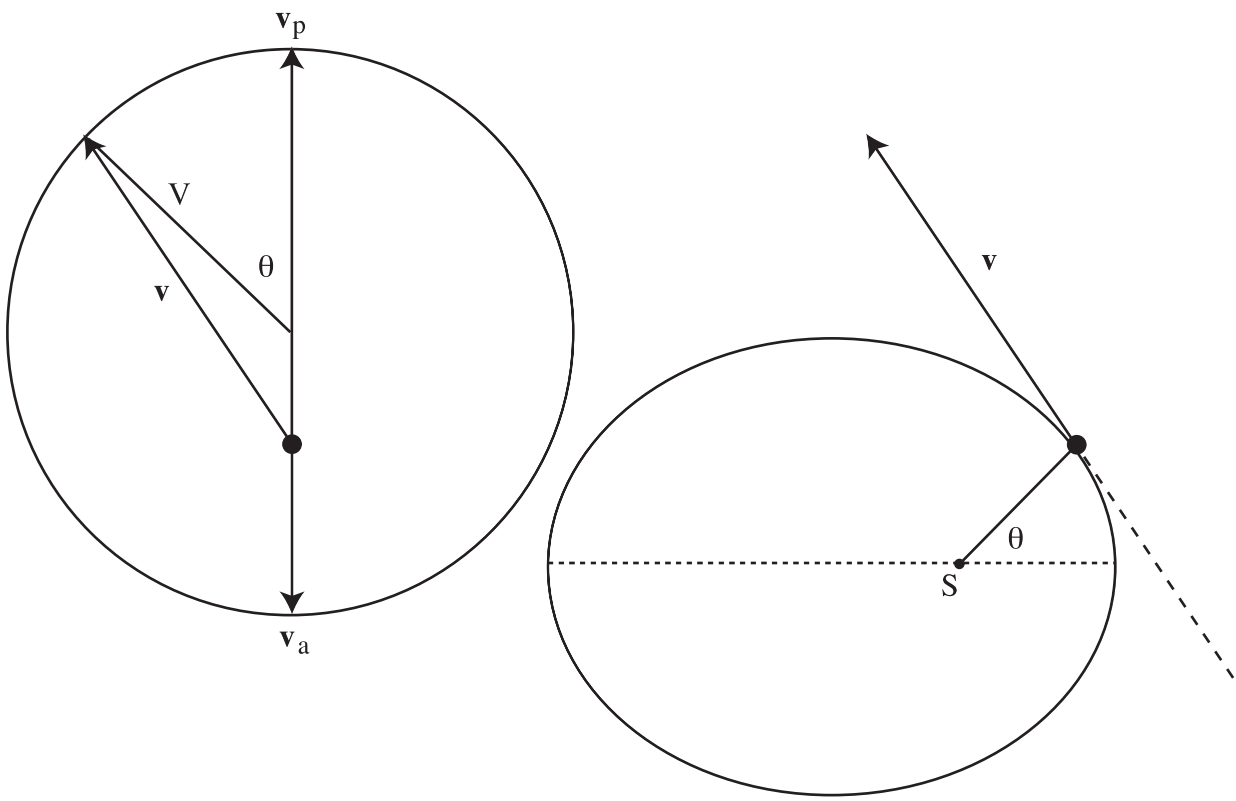

Figure 1.7: Hodograph and orbit depicting asteroid at $\theta \approx 46°$

reveals that the longest "diameter" has a length equal to that of the string afixed to the two focal points S and S'. We define the eccentricity, e, of the ellipse by taking the distance between the the focal points to be

$$|\overline{SS'}| = 2ea \tag{1.14}$$

We can derive some useful relationships between the perihelion and aphelion distances (r_p and r_a), as well as the velocities at those points, (v_p and v_a). First, we note that

$$r_a + r_p = 2a \tag{1.15}$$

From the symmetry of the focal points S and S' we can conclude

$$\begin{aligned} r_p &= \frac{1}{2}(2a - 2ea) = a(1-e) \\ r_a &= 2a - r_p = a(1+e) \end{aligned} \tag{1.16}$$

Inspecting equation 1.8 we may conclude that $h = r_a v_a = r_p v_p$ and then using 1.16 we get

$$\frac{r_a}{r_p} = \frac{v_p}{v_a} = \frac{1+e}{1-e} \tag{1.17}$$

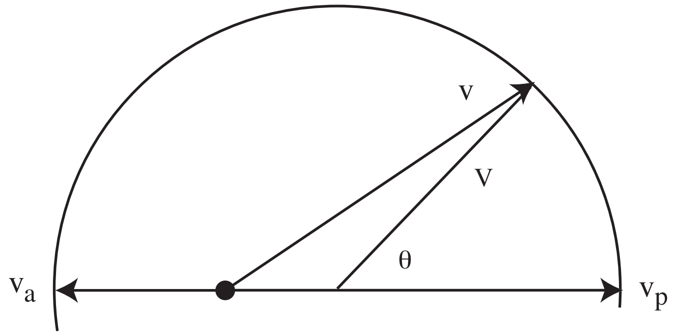

Figure 1.8: Hodograph rotated clockwise by 90°.

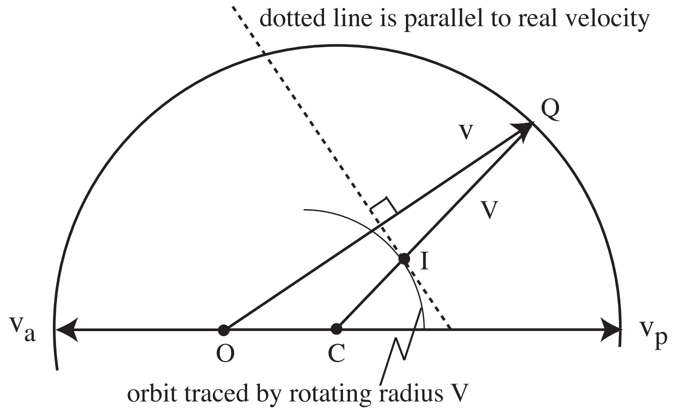

Figure 1.9: Dotted line is the perpendicular bisector of $\mathbf{v}$.

The work–energy theorem applied to the inverse square law acting leads to

$$\frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = \frac{\mu}{r_2} - \frac{\mu}{r_1} \tag{1.18}$$

where $\mu = G(M + m)$ and the subscripts 1 and 2 refer to any two points in the orbit. We may use this equation to conclude that the quantity

$$\frac{1}{2}v^2 - \frac{\mu}{r} \equiv \epsilon \tag{1.19}$$

is a constant of the motion. We call ϵ the energy per unit asteroid mass. It is relatively easy to see that for bound orbits $\epsilon < 0$.

Now let us apply equation 1.19 at the perihelion and aphelion. At either of these two points equation 1.8 gives us $rv = h$ so we can eliminate v in equation 1.19 to obtain

$$\frac{1}{2}\left(\frac{h}{r}\right)^2 - \frac{\mu}{r} = \epsilon \tag{1.20}$$

We may rearrange this equation to obtain an equation quadratic in r,

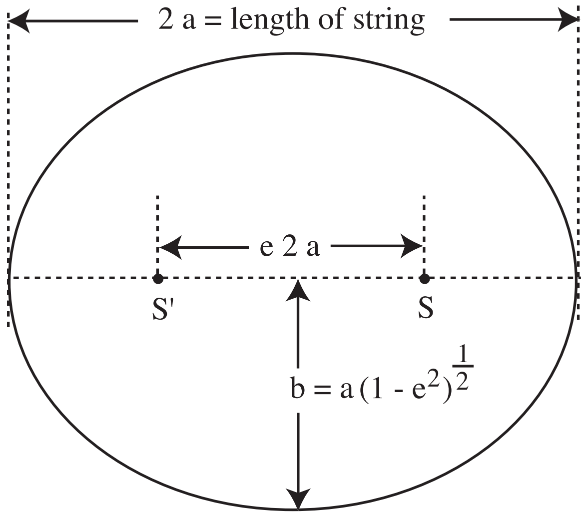

Figure 1.10: Some ellipse terminology.

$$\epsilon r^2 + \mu r - \frac{h^2}{2} = 0 \tag{1.21}$$

The two r values that satisfy this equation must be the perihelion and aphelion. Thus, we have

$$r_{p,a} = \frac{1}{2\epsilon}\left(-\mu \pm \sqrt{\mu^2 + 2h^2\epsilon}\right) \tag{1.22}$$

Adding the two roots and using equation 1.15 we get

$$r_p + r_a = 2a = -\frac{\mu}{\epsilon} \tag{1.23}$$

Thus, we find a connection between the energy per unit mass and the semi–major axis, namely

$$\epsilon = -\frac{\mu}{2a} \tag{1.24}$$

Notice that the orbital energy is independent of the eccentricity. It depends only on the total mass of the system and the semi–major axis.

1.7 Describing the orbital path

We now use the hodograph and energy conservation to obtain a description of the orbit in terms of r and θ. Figure 1.11 shows the hodograph geometry when the asteriod is at θ.

We apply the law of cosines, using $\cos\phi = -\cos\theta$, to get

$$v^2 = V^2 + \left(\frac{1}{2}(v_p - v_a)\right)^2 + 2V\frac{1}{2}(v_p - v_a)\cos\theta \tag{1.25}$$

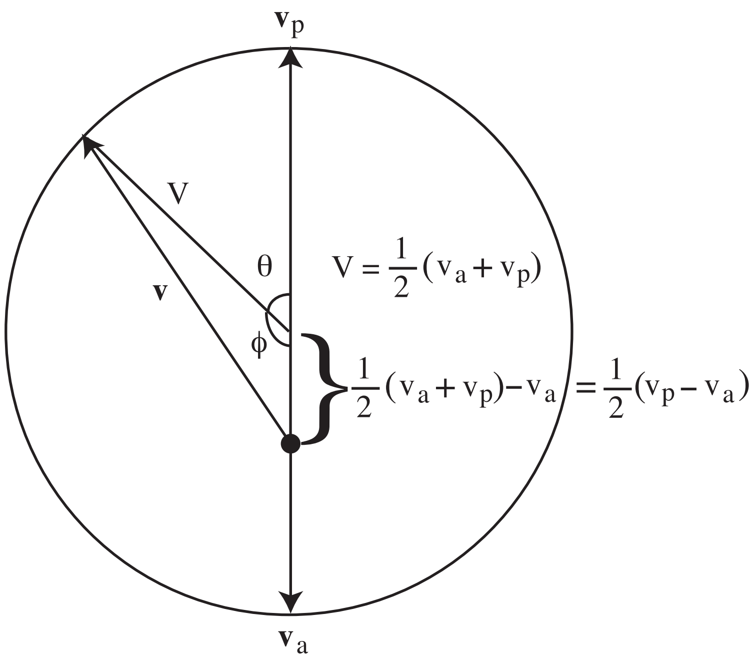

Figure 1.11: The hodograph when the asteroid is at θ.

Eliminating V using $V = \frac{1}{2}(v_p + v_a)$ we get

$$v^2 = \frac{1}{2}(v_p^2 + v_a^2) + \frac{1}{2}(v_p^2 - v_a^2)\cos\theta \tag{1.26}$$

Using energy conservation we get

$$v^2 = 2\epsilon + \frac{\mu}{r_p} + \frac{\mu}{r_a} + \left(\frac{\mu}{r_p} - \frac{\mu}{r_a}\right)\cos\theta \tag{1.27}$$

We now eliminate 2ϵ using equation 1.24 and v^2 using equation 1.19 (and 1.24) to get

$$\frac{2\mu}{r} - \frac{\mu}{a} = -\frac{\mu}{a} + \frac{\mu}{r_p} + \frac{\mu}{r_a} + \left(\frac{\mu}{r_p} - \frac{\mu}{r_a}\right)\cos\theta \tag{1.28}$$

We simplify the parenthetical terms using equation 1.16:

$$\begin{aligned} \frac{1}{r_p} + \frac{1}{r_a} &= \frac{1}{a(1-e)} + \frac{1}{a(1+e)} = \frac{2}{a(1-e^2)} \\ \frac{1}{r_p} - \frac{1}{r_a} &= \frac{1}{a(1-e)} - \frac{1}{a(1+e)} = \frac{2e}{a(1-e^2)} \end{aligned} \tag{1.29}$$

With these results equation 1.28 becomes

$$\frac{2}{r} = \frac{2}{a(1-e^2)} + \frac{2e}{a(1-e^2)}\cos\theta \tag{1.30}$$

This can be arranged to get

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{1.31}$$

This equation describes the elliptical orbit.

1.8 Kepler's Third Law

Although we shall not use it here, it is worthwhile to derive a simple connection between orbital period and semi–major axis. We begin with equation 1.21 with ϵ replaced using equation 1.24 and r replaced using equation 1.31. Note that since equation 1.21 applies only at aphelion or perihelion we must put ultimately put $\cos\theta = \pm 1$ when applying equation 1.31. Equation equation 1.21 becomes

$$-\frac{\mu}{2a} \times \frac{a^2(-e^2)^2}{(1+eu)^2} + \mu \times \frac{a(1-e^2)}{1+eu} = \frac{h^2}{2} \tag{1.32}$$

where we have defined $u \equiv \cos\theta$ for convenience and have not yet put $\cos\theta = \pm 1$. This equation can be reduced

$$\frac{h^2}{2\mu} = \frac{a(1-e^2)(1+eu)}{(1+eu)^2} - \frac{a(1-e^2)^2}{2(1+eu)^2} = \frac{a(1-e^2)}{2(1+eu)^2} \times (1+2eu+e^2) = \frac{a(1-e^2)}{2} \tag{1.33}$$

where we put $u = \pm 1$ to get the last line. So h is determined by

$$h^2 = \mu a(1-e^2) \tag{1.34}$$

On the other hand, since h is twice the constant areal velocity it must equal the area of the ellipse divided by the orbital period. The area of an ellipse is πab where $b = a\sqrt{1-e^2}$ is the semi–minor axis as shown in figure 1.10. We see then that

$$h = 2 \times \frac{\pi a^2 \sqrt{1-e^2}}{T} \tag{1.35}$$

Combining equations 1.34 and 1.35 we get

$$\frac{a^3}{T^2} = \frac{\mu}{(2\pi)^2} \tag{1.36}$$

Equation 1.36 obviously implies that ratio a^3/T^2 startHere

1.9 Dependence on time:Kepler's equation

Equation 1.31 gives us the shape of the asteroid orbit. A full description of the motion requires that we know where on this orbit the asteroid is at any given time. Let us suppose that at some time $t = \tau$ the asteroid is at perihelion. (In general, we might want $t = \tau$ to correspond to any particular position in the orbit. We choose the perihelion for the sake of simplicity.) At some later time the asteroid has the coordinates (x, y), as depicted in figure 1.12, where the Sun (S) is located at $x = y = 0$.

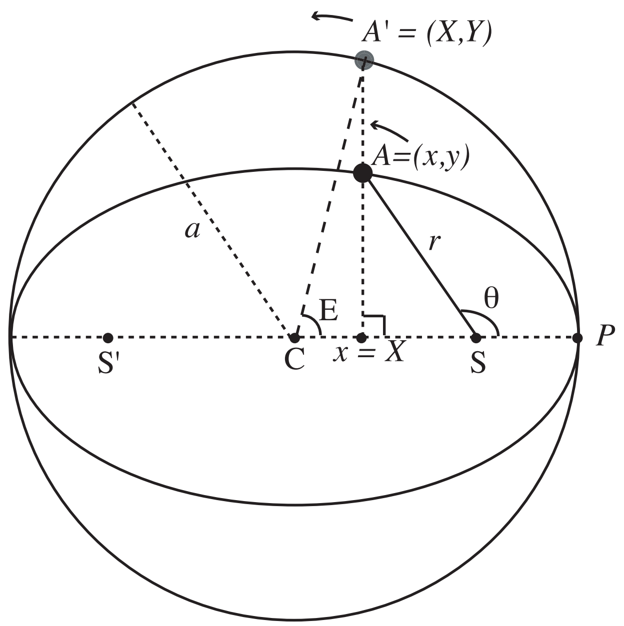

Figure 1.12: Kepler's circle.

As the asteroid A with coordinates (x, y) moves along its elliptical orbit, we imagine a ghost asteroid A' with coordinates (X, Y) that moves along a circle whose radius is equal to the semi–major axis of the ellipse. The ghost is taken to have the same x-coordinate as the asteroid at all times $(x = X)$. Let us see how Y and y are related. We will see that the relationship is surprisingly simple.

From Pythagoras' theorem $Y = \sqrt{a^2 - \overline{CX}^2}$. Since $\overline{CF} = ea$ we have $\overline{CX} = ea + r\cos\theta = ea + ru$ where we define $u \equiv \cos\theta$ for convenience. Using equation 1.31 we then have

$$\overline{CX} = ae + rae + \frac{a(1-e^2)u}{1+eu} = \frac{a(e+u)}{1+eu} \tag{1.37}$$

We then calculate

$$Y^2 = a^2 - (ae + ru)^2 = a^2 - \frac{a^2(e+u)^2}{(1+eu)^2} =$$

$$\frac{a^2}{(1+eu)^2}\left(1+2eu+e^2u^2-e^2-2eu-u^2\right)= \frac{a^2(1-e^2)(1-u^2)}{(1+eu)^2} \tag{1.38}$$

Using equation 1.31 and recognizing that $1-u^2=\sin^2\theta$ we get

$$Y=\frac{a\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}=\frac{r\sin\theta}{\sqrt{1-e^2}}=\frac{y}{\sqrt{1-e^2}} \tag{1.39}$$

Since Y and y are strictly proportional the corresponding velocities bear the same relationship so we may write

$$\frac{Y}{y}=\frac{\dot{Y}}{\dot{y}}=\frac{1}{\sqrt{1-e^2}} \tag{1.40}$$

Equation 1.40 means that the motion of the ghost asteroid A' obeys Kepler's second law since the asteroid does. We can show that as follows. The area of a parallelogram defined by the two sides consisting of the segments drawn from the origin to the coordinates (x_1, y_1) and $(x_2, y_2)$is $x_1y_2 - x_2y_1$. The rates of area sweep **about** S (called the areal velocity) of A and A' are then related:

$$\begin{aligned} &2\times \text{areal velocity of } A' = X\dot{Y}-Y\dot{X} \\ &= x\tfrac{\dot{y}}{\sqrt{1-e^2}} - y\tfrac{\dot{x}}{\sqrt{1-e^2}} = \tfrac{1}{\sqrt{1-e^2}}(x\dot{y}-y\dot{x}) \\ &= \tfrac{1}{\sqrt{1-e^2}}\times 2\times \text{areal velocity of } A \end{aligned} \tag{1.41}$$

So A' sweeps at constant areal rate the auxiliary Kepler circle of radius a. Letting T be the orbital period we can write

$$\text{rate of increase of area of sector } A'SP = \frac{\pi a^2}{T} \tag{1.42}$$

If we define the angle E (called the eccentric anomaly) as shown in figure 1.12 then we can write

$$\text{area of sector } A'SP + \text{area of triangle } CSA' = \frac{E}{2\pi}\times\pi a^2 = \frac{E}{2}\times a^2 \tag{1.43}$$

We further note the following:

$$\text{area of sector } A'SP = \frac{\pi a^2}{T}(t-\tau) \tag{1.44}$$

$$\text{area of triangle } CSA' = \frac{1}{2}\times ea\times a\sin E \tag{1.45}$$

Using these two results equation 1.43 becomes

$$\pi\left(\frac{t-\tau}{T}\right)+\frac{1}{2}e\sin E=\frac{E}{2} \tag{1.46}$$

Finally we define an angular measure of time, the mean anomaly, by $M \equiv 2\pi(t-\tau)/T$. We then obtain what is known as Kepler's equation:

$$M = E - e\sin E \tag{1.47}$$

We have an angular measure of time which is linear in time and easy calculate, the mean anomaly M. The eccentric anomaly E, is also a direct measure of time but is rather more difficult to calculate, being determined by the transcendental equation 1.47. We will find in the next section, however, that the eccentric anomaly provides a more immediate connection to orbital position

1.10 Relating position to E

Examination of figure 1.12 reveals that we can relate the Cartesian coordinates to the eccentric anomaly by

$$\begin{aligned} x &= a(\cos E - e) \\ y &= \sqrt{1-e^2}\,a\sin E \end{aligned} \tag{1.48}$$

We can then calculate $r = \sqrt{x^2+y^2}$ to get

$$r = a(1 - e\cos E) \tag{1.49}$$

We can then find $\cos\theta$ by using the above expression for x and equation 1.49 to write $\cos\theta = x/r$ as

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E} \tag{1.50}$$

We are now in a position to locate the asteroid at any time. Given an orbit in the $x-y$ plane with primary focus at the origin, the motion is fixed by three parameters which we have taken to be the semi–major axis, the eccentricity, and a time τ of perihelion passage. For any time t, we calculate the mean anomaly M and use Kepler's equation to obtain the eccentric anomaly E. Equations 1.49 and 1.50 may then be used to determine the position of the asteroid in polar coordinates. Alternatively, equations 1.48 may be used to get the asteroid position in terms of the Cartesian coordinates x and y.

In general astronomical applications, the orbit must be moved out of the $x-y$ plane into three dimensional space. Three additional angular parameters are required to decribe the orientation of the orbit so that a total of six parameters are required to specify a general orbit. In addition, as mentioned above the epoch τ may not in general refer to perihelion. As physicists sometimes do, I leave these complications to others!

Bibliography

[1] Goodstein, David L. & Judith R. Goodstein. *Feynman's Lost Lecture.* Norton, 1996.

[2] Maxwell, J. Clerk. *Matter and Motion* (1877). Dover reprint,