

# Quantum Mechanics via Dispersion—A Matrix Mechanics Primer

W. L. Andersen

July 14, 2009

## 1 Classical Dispersion Theory

Any adequate theory of atomic mechanics must describe the interaction of light with matter. We shall begin our journey by applying the laws of classical electromagnetism to a very simple model of the atom. This model proves to be remarkably useful in understanding the optical properties of the macroscopic world. Even though real atoms are of course three dimensional, in this primer we shall develop quantum mechanics in the context of atoms where each electron has only one degree of freedom. An atom is assumed to consist of  $Z$  electrons, bound harmonically (with springs if you like!) so as to have natural oscillation frequencies equal to the frequencies of the spectral lines of that atom.

Figure 1 depicts an electric wave (light) incident on such an atom. The direction of the incident light is indicated by  $\mathbf{E} \times \mathbf{B}$ , but the motion of the atomic electrons is driven primarily by the electric field of the incident light.

A periodic motion with the fundamental frequency  $\omega_0$  may be represented by the Fourier series

$$x(t) = \sum_{\tau=0}^{\infty} a_{\tau} \cos(\tau\omega_0 t + \phi_{\tau}) \quad (1)$$

or, equivalently,

$$x(t) = \sum_{\tau=-\infty}^{\infty} X_{\tau} \exp(i\tau\omega_0 t) \quad (2)$$

The  $X_{\tau}$  in equation (2) are complex and contain the phase information carried by  $\phi_{\tau}$  in equation (1). Note that in order that  $x(t)$  be real,  $X_{\tau}$  must satisfy the relation  $X_{\tau}^* = X_{-\tau}$ . Applying the Bohr-Sommerfeld condition  $\oint p dx = n\hbar$  selects certain motions or orbits which may be conveniently labeled by the integer  $n$ . No other orbits are allowed. The Fourier expansion of the  $n^{\text{th}}$  orbit can be written

$$x_n(t) = \sum_{\tau=-\infty}^{\infty} X(n; \tau) \exp(i\omega(n; \tau)t) \quad (3)$$

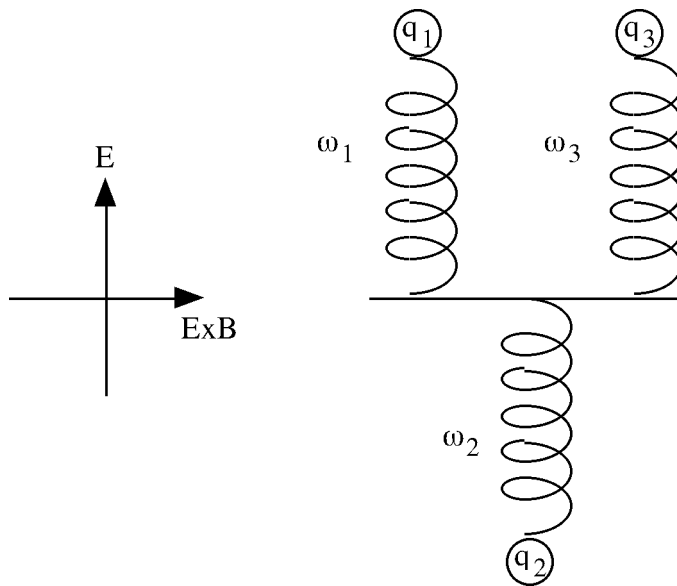


Figure 1: Light incident on an atom made of springs.

where  $\omega(n; \tau) = \tau\omega_n$  and  $\omega_n = \omega(n; 1)$  is the fundamental frequency of the  $n^{\text{th}}$  orbit. Classically, the  $\omega(n; \tau)$  are expected to be the radiation frequencies. Quantum theoretically, however, the Ritz-Rydberg combination rule leads us to expect that the radiation frequencies are given by

$$\omega(n, n - \tau) = \frac{E_n - E_{n-\tau}}{\hbar} \quad (4)$$

Note that we have two notations for frequencies. One refers to the old quantum theory based on the Bohr condition. The other refers to the as yet unrevealed quantum theory. For example,  $\omega(n; \tau)$  refers to the frequency of the  $\tau^{\text{th}}$  harmonic of the  $n^{\text{th}}$  Bohr orbit, while  $\omega(n, n - \tau)$  refers to the frequency of radiation emitted as a result of the transition  $n \rightarrow n - \tau$ . According to the Correspondence Principle, these two frequencies should agree for large  $n$ . As an exercise, the reader might use Bohr-Sommerfeld quantization to solve for the motion of a particle in a one-dimensional box and show that the frequencies in equation (4) agree with those in (1) or (2). The agreement hinges on placing a restriction as to which transitions are allowed. Such restriction is called a selection rule.

Let us consider a particular harmonic,  $\tau$ , of a particular motion,  $n$ . That is, let us consider one particular term in (1) or (2). We shall for the time being set  $\omega(n; \tau) = \omega_\tau$  and  $X(n; \tau) = X_\tau$  since we shall be working with only one Bohr orbit for a while. Quantum theoretically, considering only one orbit corresponds to restricting ourselves to the consideration of a particular pair of states,  $n$  and  $n - \tau$ , where  $n$  and  $n - \tau$  are Bohr-Sommerfeld integers. Let  $q$  be the charge

associated with this term and let  $\omega_\tau = \omega(n; \tau)$  be the angular frequency of this particular term. We shall model this term as a simple harmonic oscillator with frequency  $\omega_0$ . Suppose an electric wave of angular frequency  $\omega$  and amplitude  $E$  is incident on the system. The equation of motion for this term can be written

$$m\ddot{x}_n + m\gamma\dot{x}_n + m\omega_\tau^2 x_n = qE \cos(\omega t) \quad (5)$$

where  $m$  is the mass of the electron. The damping coefficient  $\gamma$  represents the effects of molecular collisions or any other mechanism which would rob energy from the electronic oscillation. Later we shall consider a tenuous gas and  $\gamma$  will represent the effects of radiation damping. Using either formalism (1) or (2), we may solve this equation to obtain the non-transient solution

$$x_n(t) = \frac{qE}{m} \frac{1}{\sqrt{(\omega_\tau^2 - \omega^2)^2 + (\gamma\omega)^2}} \cos(\omega t - \phi_\tau) \quad (6)$$

where  $\phi_\tau$  is the phase associated with this particular term and

$$\tan(\phi_\tau) = \frac{\gamma\omega}{\omega_\tau^2 - \omega^2} \quad (7)$$

Expressions (6) and (7) may be used to obtain the maximum dipole moment per unit volume  $P_0$  of a gas consisting of  $N$  atoms per unit volume. If we now include all harmonics and assume that  $\omega$  is not near any  $\omega_\tau$  we obtain

$$P_0 = N \sum_\tau qX_\tau = N \sum_\tau q \frac{qE}{m} \frac{1}{\sqrt{(\omega_\tau^2 - \omega^2)^2 + (\gamma\omega)^2}} \text{sign}(\omega_\tau - \omega) \quad (8)$$

When interpreting experimental data, it is customary to set  $q^2 = f_\tau e^2$ , where  $e$  is the electronic charge and  $f_\tau$  is called the oscillator strength.

$$P_0 = N \frac{e^2 E}{m} \sum_\tau \frac{f_\tau}{\sqrt{(\omega_\tau^2 - \omega^2)^2 + (\gamma\omega)^2}} \text{sign}(\omega_\tau - \omega) \quad (9)$$

Notice that, due to the phase angle  $\phi_\tau$ , each contribution to  $P$  may either enhance or oppose  $E$  depending on whether  $\omega_\tau \gg \omega$  or  $\omega_\tau \ll \omega$ .

According to elementary physical optics  $P_0$  is measurable because of its connection to the index of refraction,  $n$ . It can be shown that (See, e.g. Feynman's Lectures on Physics)

$$n = \sqrt{1 + \frac{P_0}{\epsilon_0 E}} \approx 1 + \frac{1}{2\epsilon_0} \frac{P_0}{E} \quad (10)$$

We shall now assume damping is slow enough so that  $\gamma \ll \omega$ . Substituting equation (9) into (10), we obtain

$$n = 1 + \frac{1}{2\epsilon_0} \frac{e^2 N}{m} \sum_\tau \frac{f_\tau}{\omega_\tau^2 - \omega^2} \quad (11)$$

Notice that the sign of the denominator takes care of the phase. In the high frequency limit ( $\omega \gg \omega_\tau$ ), expressions (9) and (11) become

$$P_0 = -N \frac{e^2 E}{m\omega^2} \sum_{\tau} f_{\tau} \quad (12)$$

and

$$n = 1 - \frac{N}{2\epsilon_0} \frac{e^2 E}{m\omega^2} \sum_{\tau} f_{\tau} \quad (13)$$

Equation (13) predicts an index of refraction less than one. Although (13) was derived for dilute gases, it turns out that the index of refraction of X-rays in most materials is indeed less than one. Equation (11) may be used, in conjunction with measurements of the index of refraction, to determine the oscillator strengths of atoms and molecules in a dilute gas. Using oscillator strengths determined in this fashion, equation (13) has been verified in the high frequency limit. Thus (12) may be assumed valid as well.

## 2 The Quantum-Theoretical Interpretation of the Oscillator Strengths of Classical Dispersion Theory

We consider a simple harmonic oscillator in equilibrium with cavity radiation. We shall first recall the results of classical electromagnetism. If the average energy of the oscillator is  $\langle U \rangle$ , then the average radiated power is

$$\left\langle \frac{dU}{dt} \right\rangle = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{fe^2\omega_0^2}{mc^3} \langle U \rangle \quad (14)$$

where we have used  $q^2 = fe^2$ . This formula assumes that only one degree of freedom is allowed. Notice that the quantity

$$\gamma^{-1} = 4\pi\epsilon_0 \frac{3}{2} \frac{mc^3}{e^2\omega_0^2} \quad (15)$$

may be interpreted as the decay constant of a free oscillator of charge  $e$ .

Classical electromagnetism also gives us a relationship between the energy density of radiation and the average energy of the oscillator. If we let  $\rho$  be the angular frequency distribution of cavity radiation at the frequency  $\omega_0$ . The energy balance between radiated and absorbed power gives us

$$\rho_0 = \frac{\omega_0^2}{\pi^2 c^3} \langle U_0 \rangle \quad (16)$$

We now use (16) to substitute for  $\langle U_0 \rangle$  in (14) and get

$$\left\langle \frac{dU_0}{dt} \right\rangle = \gamma_0 f_0 \frac{\pi^2 c^3}{\omega_0^2} \rho_0 \quad (17)$$

We assume that this relation holds for every pair of energy states in the Bohr atoms which are in equilibrium with the cavity radiation. Remember, each pair has an associated oscillator strength  $f$  and decay constant  $\gamma$ .

Now we consider Einstein's quantum theoretical treatment. According to this theory, each virtual oscillator absorbs cavity radiative energy at a rate

$$\left\langle \frac{dU_0}{dt} \right\rangle = B_0 \rho_0 \hbar \omega_0 \quad (18)$$

where

$$B_0 = \frac{\pi^2 c^3}{\hbar \omega_0^3} A_0 \quad (19)$$

Energy conservation demands that we set (17) equal to (18). We obtain

$$\begin{aligned} \gamma_0 f_0 \frac{\pi^2 c^3}{\omega_0^2} \rho_0 &= B_0 \rho_0 \hbar \omega_0 \\ &= \frac{\pi^2 c^3}{\hbar \omega_0^3} A_0 \rho_0 \hbar \omega_0 \end{aligned} \quad (20)$$

Finally we can solve for  $f_0$  to get

$$f_0 = \frac{A_0}{\gamma_0} \quad (21)$$

We have now related the oscillator strength  $f$ , which characterizes dispersion, to  $A$ , which characterizes spontaneous radiation.

### 3 The Correspondence Principle Applied to Radiation from Virtual Oscillators

In this section we shall demand the equivalence ( in the case of large quantum numbers ) of two different descriptions of radiative energy loss from a Bohr atom. One description will be the classical description in terms of Fourier amplitudes. The other will be the Einstein's treatment utilizing probabilities. Our considerations will yield a connection between the classical amplitudes and the probability coefficients.

We consider an isolated atom in an excited state. On the one hand, according to Einstein, the energy loss is governed by the probability coefficient  $A$  so that the average radiated power is

$$\left\langle \frac{dU_0}{dt} \right\rangle = A_0 \hbar \omega_0 \quad (22)$$

On the other hand, according to classical theory spontaneous emission is radiation from an accelerated charge. If we adopt equation (2) as a description of the motion then according to classical electrodynamics the instantaneous radiated power ( all frequencies ) is

$$\begin{aligned}
\left\langle \frac{dU}{dt} \right\rangle &= \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{e^2}{c^3} |\ddot{x}|^2 \\
&= \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{e^2}{c^3} \sum_{\alpha} \sum_{\beta} \omega_{\alpha}^2 \omega_{\beta}^2 X_{\alpha} X_{\beta} \exp[i(\omega_{\alpha} + \omega_{\beta})t] \quad (23)
\end{aligned}$$

If we take the time average we get the result

$$\left\langle \frac{dU}{dt} \right\rangle = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{e^2}{c^3} \sum_{\tau} |X_{\tau}|^2 \omega_{\tau}^4 \quad (24)$$

Let us now consider only a single Fourier component, which must correspond to a single virtual oscillator and a single spectral line. We then have

$$\left\langle \frac{dU_0}{dt} \right\rangle = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{e^2}{c^3} 2 |X_0|^2 \omega_0^4 \quad (25)$$

The extra factor of two comes from the sum over  $\tau$ , which includes positive and negative integers.

In the limit of large quantum numbers, (22) and (25) must yield the same result. If we set them equal to each other and solve for the Einstein coefficient  $A_0$  we get

$$A_0 = \frac{1}{4\pi\epsilon_0} \frac{4}{3} \frac{e^2 \omega_0^3}{c^3 \hbar} |X_0|^2 \quad (26)$$

If we choose representation (1) rather than (2), instead of (26), we obtain

$$A_0 = \frac{1}{4\pi\epsilon_0} \frac{1}{3} \frac{e^2 \omega_0^3}{c^3 \hbar} |a_0|^2 \quad (27)$$

This is a very significant result. It relates the quantum-theoretic probability coefficient  $A$  with the classical amplitude of the particular harmonic being considered. Notice that  $\hbar$  is involved in this correspondence. Expressions (26) and (27) hold only for large quantum numbers, however. The rest of our work will consist of attempting to find a quantum-theoretic replacement for  $|X|^2$  which will make (26) valid for all quantum numbers.

## 4 The Correspondence Principle Applied to Radiation Absorption by Virtual Oscillators

In this section we shall calculate the work done by radiation incident on an atom comprised of virtual oscillators. We shall see that at high frequencies the correspondence principle will be violated unless some virtual oscillators make a negative contribution to the power absorption. This shall be seen to correspond to de-excitations putting energy back into the radiation field.

Consider first a single virtual oscillator associated with  $N$  atoms per unit volume. The power absorbed is

$$\begin{aligned}
 \left\langle \frac{dU_0}{dt} \right\rangle &= N \langle q \dot{x} E \cos(\omega t) \rangle \\
 &= N q \omega \frac{q E}{m} \frac{1}{\omega_0^2 - \omega^2} E \langle \cos^2(\omega t) \rangle \\
 &= N \frac{q^2 E^2}{2m} \frac{\omega}{\omega_0^2 - \omega^2}
 \end{aligned} \tag{28}$$

If we consider the high frequency response ( $\omega \gg \omega_0$ ) of virtual oscillators which are initially in the  $n^{\text{th}}$  orbit then the absorbed power is

$$\left\langle \frac{dU}{dt} \right\rangle = -N \frac{e^2 E^2}{2m\omega} \sum_{\tau} f_{\tau} = -N \frac{e^2 E^2}{2m\omega} \sum_{\tau} \frac{A_{\tau}}{\gamma_{\tau}} \tag{29}$$

Let us now consider the special case of an atom consisting only of one virtual oscillator, i.e. an electron on a single spring. Let us suppose our simple harmonic atom to be in an excited state  $n > 0$ . Then according to (27), we have

$$A_0 = \frac{1}{4\pi\epsilon_0} \frac{1}{3} \frac{e^2 \omega_0^3}{c^3 \hbar} |a_0|^2 \tag{30}$$

We may use Bohr-Sommerfeld quantization to get the Einstein coefficient. Also, since the correspondence principle only allows one downward transition ( $n \rightarrow n - 1$ ) we may relabel  $A_0 \rightarrow A_n$  and  $a_0 \rightarrow a_n$ . Thus

$$a_n = a_{max} = \sqrt{\frac{2E_n}{m\omega_0^2}} = \sqrt{\frac{2n\hbar}{m\omega_0}} \tag{31}$$

Substituting (31) into (30) we get

$$A_0 = \frac{1}{4\pi\epsilon_0} \frac{1}{3} \frac{e^2 \omega_0^3}{c^3 \hbar} \frac{2n\hbar}{m\omega_0} = n\gamma_n \tag{32}$$

We have just calculated the oscillator strengths for a quantum harmonic oscillator! They are

$$f_n = \frac{A_n}{\gamma_n} = n \tag{33}$$

We know from Planck's work that this atom may absorb or emit only one quantity of energy, namely  $E = \hbar\omega_0$ . Therefore, the sum in (29) contains only two terms, namely  $n \rightarrow n - 1$  and  $n \rightarrow n + 1$ . Using (33) equation (29) becomes

$$\left\langle \frac{dU}{dt} \right\rangle = -N \frac{e^2 E^2}{2m\omega} (n + n + 1) = -N \frac{e^2 E^2}{2m\omega} (2n + 1) \tag{34}$$

Thus, according to our calculation, it seems that the rate of energy absorption increases as the oscillator's energy increases. This contradicts classical theory according to which we must have

$$\left\langle \frac{dU}{dt} \right\rangle = -N \frac{e^2 E^2}{2m\omega} \quad (35)$$

which is independent of  $n$ . Since (34) and (35) disagree for large  $n$ , the correspondence principle is violated. Let's look more carefully at (34).

In (34) we have essentially added the emission and absorption oscillator strengths together. Since absorption increases  $U$  and emission decreases  $U$ , it seems reasonable to suppose that we should instead add the absorption  $f$  and subtract the emission  $f$ . In other words, we should assign negative values to emission oscillator strengths. If we take this approach, (34) becomes

$$\left\langle \frac{dU}{dt} \right\rangle = -N \frac{e^2 E^2}{2m\omega} (n + 1 - n) = -N \frac{e^2 E^2}{2m\omega} \quad (36)$$

Thus the lesson we have learned is, when summing over oscillator strengths we should write

$$\sum_{\tau} f_{\tau} = \sum \frac{A_a}{\gamma_a} - \sum \frac{A_e}{\gamma_e} \quad (37)$$

where the subscripts refer to absorption and emission processes. The first term, representing absorption, involves a sum over all  $n$  greater than the initial orbit of the atom. The second term, representing emission, involves a sum over all  $n$  less than the initial orbit of the atom. Although (37) was "derived" for a simple harmonic oscillator, it can be shown that the principle of correspondence requires (37) for any force law binding the electron.

## 5 The Correspondence Principle Leads to the Algebra of Quantum Dynamical Variables

For equations 12 and (37) to hold for atomic systems in the high frequency limit we must have

$$\sum_{\tau} f_{\tau} = \sum \frac{A_a}{\gamma_a} - \sum \frac{A_e}{\gamma_e} = 1 \quad (38)$$

for the case of a single bound electron. Using (15) and (25) we get

$$\begin{aligned} \sum \frac{A_a}{\gamma_a} - \sum \frac{A_e}{\gamma_e} &= 4\pi\epsilon_0 \frac{3}{2} \frac{mc^3}{e^2} \left( \sum \frac{A_a}{\omega_a^2} - \sum \frac{A_e}{\omega_e^2} \right) \\ &= \frac{2m}{\hbar} \left( \sum \omega_a |X_a|^2 - \sum \omega_e |X_e|^2 \right) = 1 \end{aligned} \quad (39)$$

We have adopted representation (2) because the simple properties of complex multiplication will prove useful later. Choosing representation (1) leads to



$$\sum \frac{A_a}{\gamma_a} - \sum \frac{A_e}{\gamma_e} = \frac{m}{2\hbar} (\sum \omega_a |a_a|^2 - \sum \omega_e |a_e|^2) = 1 \quad (40)$$

The final transition to modern quantum mechanics requires a non-deductive, albeit well motivated, step. Each  $X_a$  and  $X_e$  is associated with a pair of quantum orbits. The corresponding frequencies,  $\omega_a$  and  $\omega_e$ , may each be labeled by two integers identifying the initial and final orbits. There is complete symmetry between these integers in that each label orbits. In the classical interpretation, however, each frequency is to be labeled by an integer  $n$ , identifying the initial orbit, and an integer  $\tau$  which specifies the particular harmonic of the  $n^{\text{th}}$  orbit.

Thus there is a certain symmetry which is lost in the classical interpretation which associates each virtual oscillator with a motion, labeled by a integral quantum number, and a harmonic, labeled by another integer which is not a quantum number. Let us therefore adopt a more symmetric interpretation and notation. We define

$$\omega(n + \tau, n) = \frac{E_{n+\tau} - E_n}{\hbar} \quad (41)$$

which is an absorption frequency if  $\tau > 0$  and is an emission frequency if  $\tau < 0$ . This is just equation (4). Similarly, using the real representation of equation (1), we define

$$a_a = a(n + \tau, n) \quad \text{and} \quad a_e = a(n, n - \tau) \quad (42)$$

If we choose the representation (2) we write

$$X_a = X(n + \tau, n) \quad \text{and} \quad X_e = X(n, n - \tau) \quad (43)$$

With these notations, (39) and (40) become

$$\sum_{\tau=0}^{\infty} [\omega(n + \tau, n) |a(n, n + \tau)|^2 - \omega(n, n - \tau) |a(n, n - \tau)|^2] = \frac{2\hbar}{m} \quad (44)$$

$$\sum_{\tau=0}^{\infty} [\omega(n + \tau, n) |X(n, n + \tau)|^2 - \omega(n, n - \tau) |X(n, n - \tau)|^2] = \frac{\hbar}{2m} \quad (45)$$

These last two expressions contain the essence of quantum mechanics. In particular, we shall now see that (45) leads directly to the Heisenberg commutation or exchange relation.

First, we note that

$$\begin{aligned} \sum_{\tau=-\infty}^0 \omega(n + \tau, n) |X(n, n + \tau)|^2 &= - \sum_0^{\infty} \omega(n, n - \tau) |X(n, n - \tau)|^2 \\ \sum_{\tau=-\infty}^0 \omega(n - \tau, n) |X(n, n - \tau)|^2 &= - \sum_0^{\infty} \omega(n, n + \tau) |X(n, n + \tau)|^2 \end{aligned} \quad (46)$$

Using (46) and the identity  $|X(\alpha, \beta)|^2 = X(\alpha, \beta)X(\beta, \alpha)$  we may write (45) as

$$\sum_{\tau=-\infty}^{\infty} [\omega(n + \tau, n)X(n + \tau, n)X(n, n + \tau) - \omega(n, n - \tau)X(n, n - \tau)X(n - \tau, n)] = \frac{\hbar}{m} \quad (47)$$

We represent the momentum  $p = m\dot{x}$  by

$$P(\alpha, \beta) = im\omega(\alpha, \beta)X(\alpha, \beta) \exp[i\omega(\alpha, \beta)t] \quad (48)$$

If we use (48) then (47) becomes the Heisenberg commutation relation,

$$\sum_{\tau=-\infty}^{\infty} [X(n, n + \tau)P(n + \tau, n) - P(n, n - \tau)X(n - \tau, n)] = i\hbar \quad (49)$$

In obtaining (49) we have used

$$X(\alpha, \beta)X(\beta, \alpha) = X(\alpha, \beta) \exp[i\omega(\alpha, \beta)t]X(\beta, \alpha) \exp[i\omega(\beta, \alpha)t] \quad (50)$$

which follows from  $\omega(\alpha, \beta) + \omega(\beta, \alpha) = \omega(\alpha, \alpha) = 0$  and the simple properties of the products of exponentials. Such a pleasant identity as (50) does not hold for the  $a(\alpha, \beta)$  coefficients of representation (1). This is our reason for preferring representation (2).

Heisenberg's exchange relation (49) along with Newton's laws of motion supply a complete (non-relativistic) quantum theoretical description of any mechanical system. For example, the equation of motion of a simple harmonic oscillator with force constant  $k$  in the quantum theory is simply

$$m\ddot{x} = -kx \quad (51)$$

Thus the equation of motion in the quantum theory has exactly the form has in classical theory. The only difference is that the condition (49) is to be imposed on the dynamical variables  $x$  and  $p$ .

Although the matrix theory is often difficult to calculate with and can not be readily applied to unbound systems, it has the advantage that its development follows the experimental evidence in a more detailed fashion than wave mechanics. As Victor Weisskopf once said of the early quantum mechanics, "We calculated with Schrödinger, but we believed with Heisenberg."

Although these notes are very rough and certainly too brief, I hope this little tract will inspire you to continue grappling with quantum mechanics, and perhaps even be of help as you try to develop an intuition for this strange and beautiful aspect of the natural world. Unfortunately, there are few references for those interested in exploring the transition from classical to quantum mechanics as motivated by the correspondence principle. *From c-Numbers to*

*q-Numbers* by Olivier Darrigol is the closest thing to a text book in print [1]. However, the best single reference I know of for further work along these lines is B. L. van der Waerden's collection of reprints, *Sources of Quantum Mechanics* [2]. These English translations in fact form the basis of these notes, which are little more than an abbreviated version of the story told there.

## References

- [1] Olivier Darrigol. *From c-Numbers to q-Numbers: The Classical Analogy in the History of Quantum Theory*. University of California Press, 1992.
- [2] B. L. van der Waerden. *Sources of Quantum Mechanics*. Dover, 1968.